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Induced Drag Ideal Efficiency Factor of Arbitrary Lateral-Vertical Wing Forms

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Induced Drag Ideal Efficiency Factor of Arbitrary Lateral-Vertical Wing Forms

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SUMMARY

A relatively simple equation is presented for estimating the induced drag ideal efficiency factor e for arbitrary cross-sectional wing forms. This equation is based on eight basic but varied wing configurations which have exact solutions. The e function which relates the basic wings is developed statistically and is a continuous function of configuration geometry. The basic wing configurations include boxwings shaped as a rectangle, ellipse, and diamond; the V-wing; end-plate wing; 90 degree cruciform; circle dumbbell; and biplane. Example applications of the e equations are made to many wing forms such as wings with struts which form partial span rectangle, diamond, triangle, ellipse, and semi-ellipse wings; the ellipse and rectangle dumbbell wings; bow-tie, cruciform, winglet, and fan wings; and multi-wings. Derivations are presented in the appendices of exact closed form solutions found of e for the V-wing and 90 degree cruciform wing and for an asymptotic solution for multi-wings.

INTRODUCTION

With the objective of better economics through improved aerodynamic and structural shapes, many unconventional aircraft configurations are being investigated. Particularly desirable are wing forms for which both drag and structural stresses decrease. A measure of aerodynamics efficiency is the induced drag ideal efficiency factor denoted by e.

Historically, exact solutions for induced drag have been found for several nonplanar wing configurations. These include the rectangle boxwing, end-plate wing, and biplane reported in reference 1; the ellipse boxwing and circle dumbbell wing from the methods of reference 2; and the diamond boxwing and V-wing in reference 3. These solutions form the basis for the present study.

The objective of this study is to develop a generalized equation for estimating the ideal e for arbitrary yz-plane wing forms. For a given wing form the ideal e is the maximum that e can become, and at ideal e all the wing form surfaces have ideal loading distributions such that the total induced drag is minimized. A generalized equation for estimating e provides the designer with

a tool for evaluating the aerodynamic efficiency for the configuration under consideration.

SYMBOLS

b	lateral span along y-axis of wing configuration
D_i	induced drag, equation (1)
E	complete elliptic integral of the second-kind, with modulus k
e	induced drag ideal efficiency factor, equation (1)
e_e	e for end-plate wing, equation (16)
e_v	e for V-wing, equation (15)
f	function of parameter P, equation (20)
G_0	function of $\eta_{10}H$, equation (18)
G_1	function of e_v, e_e, η_{af}, equation (19)
H	dimensionless maximum height, h_m/b
H_a	average dimensionless height, equation (25)
h_m	maximum height of wing configuration
K	complete elliptic integral of the first kind, with modulus k
k	modulus for K and E, $k = \sin\alpha$ where α is the modular angle
L	total lift of wing system, equation (1)
N	total number of wings
P	parameter, equation (21)
p	perimeter of wing system, equation (29)
p_e	perimeter of ellipse, equation (22)
q	free stream dynamic pressure, $\frac{1}{2}\rho V^2$

y	lateral coordinate, zero at midspan, positive to right
Γ	gamma function, equation (14)
γ	dihedral angle
η	dimensionless lateral coordinate, 2y/b
η_a	averaged η of outboard face of wing, equations (28) and (76)
η_{10}	span station of difference, equation (27)
η_{af}	η_a for fin wings, equations (30), (31), (32)
η_0, η_1	span stations, equation (27)
η_n	span station of multi-wings, equation (76)
ρ	air density
σ	perimeter parameter, equations (29) and (77)
σ_e	perimeter parameter of ellipse, equations (22) through (24)
σ_M	perimeter parameter of multi-wing, equation (78)

IDEAL EFFICIENCY FOR BASIC CONFIGURATIONS

The present analysis is based on a group of varied wing configurations which have exact solutions for predicting e. These include the wings described as rectangle boxwing, ellipse boxwing, diamond boxwing, V-wing, end-plate wing, 90 degree cruciform wing, circle dumbbell wing, and biplane. The induced drag ideal efficiency factor listed throughout the paper relates the total induced drag to the total lift of the wing system by the equation:

$$D_i = \frac{L^2}{\pi q b^2 e} \tag{1}$$

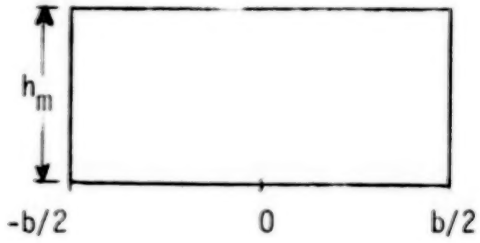
where b is the maximum span in the lateral plane, q is the dynamic pressure, and L is the sum of all the lifts, including wing, winglets, end-plates, struts, or tail. For ideal e, the lifts and loadings will have ideal distributions.

Rectangle and Ellipse Boxwings

From reference 1, p. 205, for the rectangle boxwing, the equations for the dimensionless maximum height for H and for e can be developed.

$$H = \frac{h_m}{b} = \frac{E - (1 - k^2)K}{E' - k^2K'} \qquad (2)$$

$$e = \frac{1 - k^2}{(E' - k^2K')^2} \qquad (3)$$



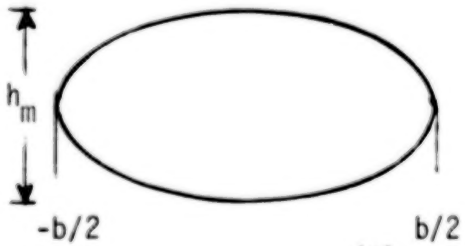
where K and E are complete elliptic integrals of the first and second kinds, respectively, with modulus k, while K' and E' are the complementary integrals. That is, K' is K with modulus $\sqrt{1 - k^2}$, and E' is E with modulus $\sqrt{1 - k^2}$. The modulus k is related to the modular angle α by $k = \sin\alpha$. Equations (2) and (3) do not permit a direct solution of e for a specified value of H. A value of k is specified and e from equation (3) is determined, then the corresponding H from equation (2) is determined using the same k value. Geometric characteristics of the rectangle include the nondimensional perimeter given by

$$\sigma = \frac{p}{2b} = \frac{2b + 2h_m}{2b} = 1 + H \qquad (4)$$

The rectangle averaged cross-sectional area per wing span squared is

$$H_a = \frac{h_m b}{b^2} = H \qquad (5)$$

The e for the ellipse boxwing can be determined from the work given in reference 2. The results are



$$e = 1 + H \qquad (6)$$

The perimeter parameter of the ellipse is an elliptic integral of the second kind given by

$$\sigma_e = \begin{bmatrix} E, & k = \sqrt{1 - H^2}, & H \leq 1 \\ HE, & k = \sqrt{1 - H^{-2}}, & H \geq 1 \end{bmatrix} \qquad (7)$$

The ellipse averaged cross-sectional area per wing span squared is

$$H_a = \frac{\pi h_m b}{4 b^2} = \frac{\pi}{4} H \tag{8}$$

Relating Rectangle and Ellipse Boxwings

Examination of equations (2) and (3) shows that at $H = 0$, $e = 1$, and at $H \to \infty$, $e \to 4H/\pi$. Then for the rectangle boxwing, e can be expressed as

$$e = (1 + \frac{4}{\pi} H)(1 + f) \tag{9}$$

where f is a function of H to be determined. Equation (9) can apply to both rectangle and ellipse boxwings by letting $H = H_a$, and f be a function of perimeter difference. Then

$$\left.\begin{aligned} e &= (1 + \frac{4}{\pi} H_a)(1 + f) \\ f &= 1.4512\ P^{.841}\ (1 - \sqrt{P})^{1.0795} \left[1 + .585\ P^3\ (P - .37)^3\right] \\ P &= \sigma - \sigma_e \end{aligned}\right\} \tag{10}$$

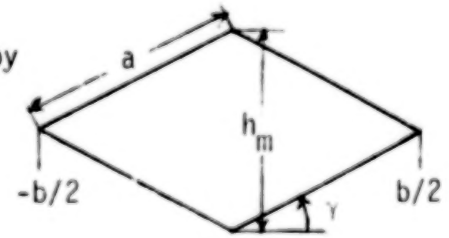
where σ_e is given in equation (7), and σ for the rectangle is given in equation (4). The f function is statistically developed from the e and H data of equations (2) and (3). Equation (10) applies to either the rectangle or ellipse boxwing and gives an estimate of e for any shape between these two. Example values of e are listed in table 1.

Diamond Boxwing

From reference 3, the added mass M is given by

$$M = 2\pi\rho\alpha$$

where (here γ replaces β of ref. 3)



$$\alpha = \frac{1}{\pi} (\frac{\pi}{2} - \gamma), \qquad \gamma = \tan^{-1} H \tag{11}$$

Then the dimensionless added mass is

$$\frac{M}{\rho\pi a^2} = \frac{1 - \frac{2}{\pi}\gamma}{a^2} = \frac{\pi(1 - \frac{2\gamma}{\pi})}{\left[\Gamma(\frac{1}{2} + \frac{\gamma}{\pi})\ \Gamma(1 - \frac{\gamma}{\pi})\right]^2} = \frac{\pi(1 - \frac{2}{\pi}\tan^{-1} H)}{\left[\Gamma(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} H)\ \Gamma(1 - \frac{1}{\pi}\tan^{-1} H)\right]^2} \qquad (12)$$

where a^2 is derived in reference 3, and $H = h_m/b$. In reference 3, D_i and L are related by

$$D_i = \frac{L^2}{2MV^2} = \frac{L^2}{\pi q b^2 \frac{4a^2}{b^2}(\frac{M}{\rho\pi a^2})} \quad . \qquad (13)$$

where from the sketch, $4a^2/b^2 = 1 + H^2$, then

$$e = (1 + H^2)(\frac{M}{\rho\pi a^2}) = \frac{\pi(1 + H^2)(1 - \frac{2}{\pi}\tan^{-1} H)}{\left[\Gamma(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} H)\ \Gamma(1 - \frac{1}{\pi}\tan^{-1} H)\right]^2} \qquad (14)$$

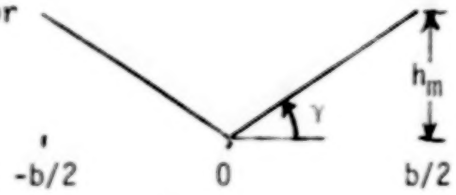
where Γ is the gamma function. For the diamond boxwing, values of e for various values of H are presented in table 1. Tables of the gamma function are given in reference 4.

TABLE 1. - IDEAL e VALUES FOR RECTANGLE, ELLIPSE, AND DIAMOND BOXWINGS

H	rectangle eq. (10)	ellipse eq. (6)	diamond eq. (14)	H	rectangle eq. (10)	ellipse eq. (6)	diamond eq. (14)
0	1	1	1	.8	2.47997	1.8	1.29518
.05	1.15179	1.05	1.01270	1.0	2.78642	2.0	1.39320
.10	1.26814	1.10	1.02622	1.2	3.08606	2.2	1.49715
.15	1.37327	1.15	1.04056	1.4	3.38041	2.4	1.60562
.20	1.47189	1.20	1.05573	1.6	3.67066	2.6	1.71751
.25	1.56610	1.25	1.07171	1.8	3.96086	2.8	1.83206
.3	1.65709	1.3	1.08850	2	4.24189	3	1.94866
.4	1.83204	1.4	1.12442	3	5.63505	4	2.55051
.5	2.00026	1.5	1.16330	4	6.99909	5	3.16840
.6	2.16362	1.6	1.20490	∞	$\to 4H/\pi$	$\to H$	$\to 2H/\pi$

V-Wing

An exact closed function solution was found for the e of a V-wing. The derivation is given in appendix A. From equation (A15)

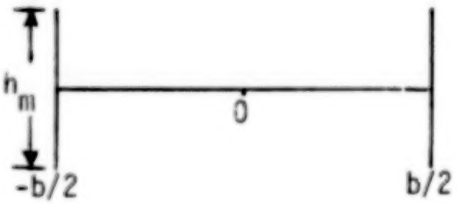


$$e_v = \frac{1}{\cos^2 \gamma} \left(\frac{1 - \frac{2\gamma}{\pi}}{1 + \frac{2\gamma}{\pi}} \right)^{\frac{2\gamma}{\pi}}, \quad \gamma = \tan^{-1} 2H \tag{15}$$

where γ is the dihedral angle in radians. Values of e for various H values are listed in table 2.

End-Plate Wing

An accurate but simplified e function for end plate wings is developed in appendix B. From equation (B14)

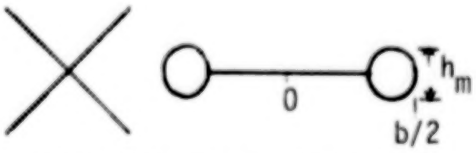


$$e_e = 1 + \left[\frac{4}{\pi} + \frac{2 - \frac{4}{\pi}}{1 + .732\,H - \frac{.0612\,H(.8 - H)^2}{1 + .3688\,H^2}} \right] H \tag{16}$$

Values of e for the end-plate wing computed from equation (16) are presented in table 2.

90° Cruciform Wing and Circle Dumbbell Wing

The 90° cruciform wing has H = 1, and as shown in appendix C, e = 2 for the unbanked wing. Also shown in appendix C is the variation of e with bank angle, and that when induced drag is taken as relative to wing length (and not lateral span) the e is unity at all bank angles.



The e for the circle dumbbell wing can be developed from induced drag methods presented in reference 2.

GENERALIZED EQUATIONS FOR ESTIMATING e

The objective is to develop an e function, with continuity, which becomes the proper e function of each of the basic configurations when that configuration geometry is inserted. Because there are many basic configurations and the e function is continuous, the application to general configurations should be within an acceptable order of accuracy. The general e function is statistically developed so that each basic configuration is taken into account.

$$e = \left\{1 + \frac{4}{\pi} H_a \left[1 - \left(1 - \frac{16}{\pi^2} \eta_a \frac{H_a}{H}\right)\left(1 - \eta_a \frac{H_a}{H}\right) G_0\right]\right\} (1 + f) + \left[1 - \left(\frac{H_a}{\eta_a H}\right)^{1.7}\right] G_1 \tag{17}$$

$$G_0 = \frac{(1 - .19\, \eta_{10} H)(1 + .4826\, \eta_{10}^2 H^2)}{(1 + 1.81\, \eta_{10} H)(1 + .255\, \eta_{10}^{5/2} H^{5/2})} \tag{18}$$

$$G_1 = \frac{(6 - 4\sqrt{2})\,(e_e - 1)\,(e_v - 1)\,\eta_{af}^2}{\left[\sqrt{e_e - 1} - \sqrt{2}\,\sqrt{e_v - 1} - (\sqrt{e_e - 1} - 2\sqrt{e_v - 1})\,\sqrt{\eta_{af}}\right]^2}$$

$$\eta_{af} = 1 \quad e_e - 1 \qquad \eta_{af} = 1/2 \quad e_v - 1 \tag{19}$$

where e_v and e_e are given in equations (15) and (16) and in table 2.

$$f = 1.4512 \frac{P}{|P|} |P|^{.841} (1 - |P|^{1/2})^{1.0795} \left[1 + .585|P|^3(|P| - .37)^3\right] \tag{20}$$

$$P = \left[\frac{\eta_{10} \frac{H_a}{H}}{1 + 3(1 - \eta_{10} \frac{H_a}{H})(1 - 2\, \eta_{10} \frac{H_a}{H})^2}\right]^{3/2} (\sigma - \sigma_e) \tag{21}$$

$$\sigma_e = \frac{\text{perimeter of full or of partial span symmetric ellipse boxwing}}{\text{twice the wing span}}$$

$$\frac{\sigma_e}{\eta_1} = \begin{cases} E, & k = \sqrt{1 - (\frac{H}{\eta_1})^2}, \quad \frac{H}{\eta_1} \leq 1 \\ \frac{HE}{\eta_1}, & k = \sqrt{1 - (\frac{H}{\eta_1})^{-2}}, \quad \frac{H}{\eta_1} \geq 1 \end{cases} \qquad (22)$$

Also functionally

$$\sigma_e(\frac{H}{\eta_1}) = \frac{H}{\eta_1}\,\sigma_e(\frac{\eta_1}{H})$$

Numerical values of σ_e/η_1 are presented in table 3.

An approximation for σ_e is

$$\frac{\sigma_e}{\eta_1} \approx \left[1 + \frac{(\frac{H}{\eta_1})^{\frac{2}{1.11374}}}{1 + (\frac{H}{\eta_1})^{\frac{1}{1.11374}}}\right]^{1.11374} \left[1 + \frac{(\frac{H}{\eta_1})^2(1 - \frac{H}{\eta_1})^2}{32\,(1 + \frac{H}{\eta_1})^6}\right] \qquad (23)$$

For a semi-ellipse in a η_1-η_0 interval on the wing semi-span

$$\sigma_e = \eta_0 + (\eta_1 - \eta_0) \begin{cases} E, & k = \sqrt{1 - (\frac{H}{\eta_1 - \eta_0})^2} \\ \frac{HE}{\eta_1 - \eta_0}, & k = \sqrt{1 - (\frac{H}{\eta_1 - \eta_0})^{-2}} \end{cases} \qquad (24)$$

$$H_a = \frac{\text{cross-section area of boxwing}}{b^2} \qquad (25)$$

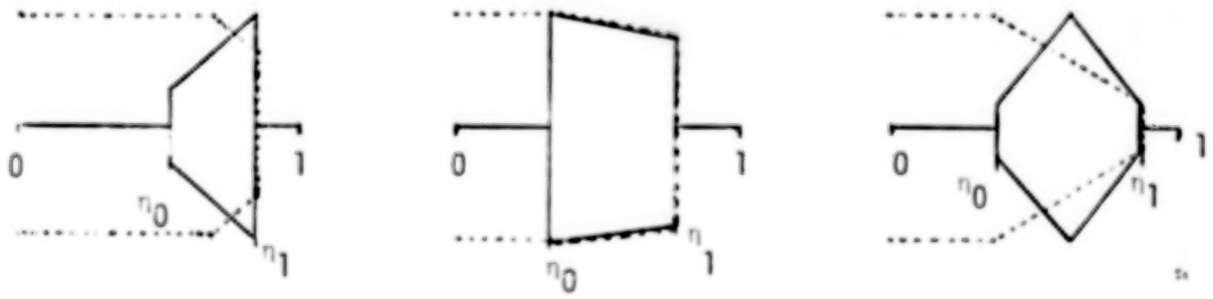
$H = \frac{h_m}{b}$, where h_m is the maximum height of wing configuration. (26)

$\eta_{10} = \eta_1 - \eta_0$, where η_1 and η_0 are the spanwise stations of the outboard and inboard sides of the boxwing at the wing. (27)

$\eta_a = \int_0^1 \eta d(\frac{h}{h_m})$, averaged η of outboard face of wing. (28)

$\sigma = \frac{p}{2b}$, where p is perimeter of full or of partial span boxwing in which h is constant or decreases with η. (29)

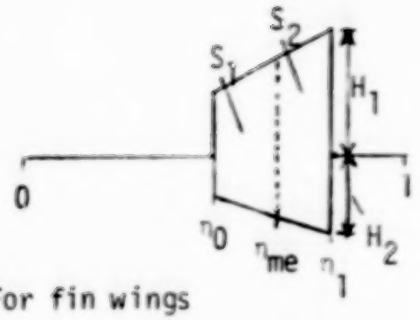
For bow-tie and dumbbell type wing configurations, the effective perimeter is determined from the shape with the maximum dumbbell height translated one-half dumbbell width towards the inboard side but limited to the inboard side of the dumbbell which is η_0. Some examples are



where ..... is the translated shape needed to determine p and σ for this type of wing. The value of σ_e of the dumbbell wing with symmetrical ellipses is given in equation (24). This σ_e and the dumbbell type wing value of σ are used in equation (21) for evaluating P.

$\eta_{af} = \eta_a$, for full or for partial span boxwings in which h is constant or decreases with η. (30)

$$\eta_{af} = \frac{1 + .05 \frac{H_2}{H_1} \eta_{me}^2}{1 + .05 \eta_{me}^2} \eta_a$$, for bow-tie, dumbbell type, and vertical fin wing configurations, where, (31)



, η_{me} is mean lateral span station for $S_1 = S_2$. For a vertical fin wing, $\eta_{me} = \eta_0 = \eta_1 = \eta_a$.

For fin wings

$$\eta_{af} = \left\{1 + \frac{1}{2}(2\eta_{a2} - \eta_{01})^2 \sqrt{\frac{H_2^2 + (\eta_{12} - \eta_{01})^2}{H_1^2 + (\eta_{11} - \eta_{01})^2}} \left(1 - \frac{\sin \gamma_2}{\sin \gamma_1}\right)\right.$$

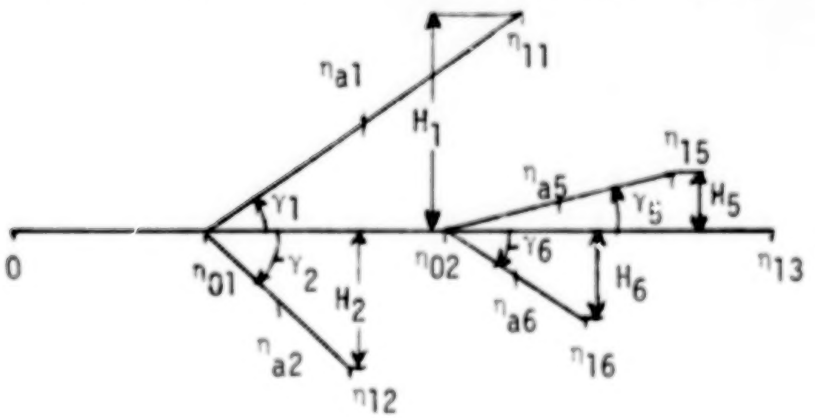
$$\left[.05 + \frac{(\eta_{11} - \eta_{01})|\eta_{11} - \eta_{01}|^{1/2}}{2.1527}\right] + \frac{1}{9}\frac{.77 + .9H}{1 + .67H}(\eta_{13} - \eta_{01})$$

$$\left.(\eta_{11} + \eta_{12} - 2\eta_{01})\right\} \frac{\eta_{a1}}{1 + .05\,\eta_{01}^2} + \frac{.77 + .9H}{1 + .67H} \sum_{n=3}^{N}$$

$$\left\{\frac{H_{2n-1} + H_{2n}}{1.15\,N\,H_1}(\eta_{a,2n} - \eta_{01}) \sqrt{\frac{H_{2n}^2 + (\eta_{1,2n} - \eta_{0,n-1})^2}{H_{2n-1}^2 + (\eta_{1,2n-1} - \eta_{a,n-1})^2}}\right.$$

$$\left.\left(1 - \frac{\sin \gamma_{2n}}{\sin \gamma_{2n-1}}\right)\left[.05 + \frac{(\eta_{1,2n-1} - .359\,\eta_{0,n-1})^{3/2}}{2.1527}\right]\right\} \eta_{a,2n-1} \qquad (32)$$

where the variables are defined through the following fin wing semispan sketch:



$\eta_{13} = 1$, $\gamma_3 = \gamma_4 = 0$

$H_3 = H_4 = 0$

TABLE 2. - IDEAL e OF V-WING AND END-PLATE WING

π	e_v V-wing eq. (15)	e_e end-plate eq. (16)	H	e_v V-wing eq. (15)	e_e end-plate eq. (16)
0	1	1	.8	1.32715	2.38527
.05	1.00189	1.09878	1.0	1.45265	2.69328
.10	1.00750	1.19523	1.2	1.58206	2.99408
.15	1.01665	1.28888	1.4	1.71362	3.28952
.20	1.02909	1.38192	1.6	1.84636	3.58082
.25	1.04452	1.47248	1.8	1.97974	3.86878
.3	1.06257	1.56140	2	2.11345	4.15394
.4	1.10524	1.73481	3	2.78235	5.5489
.5	1.15470	1.90313	4	3.44833	6.9085
.6	1.20903	2.06720	∞	$\rightarrow 2H/\pi$	$\rightarrow 4H/\pi$

TABLE 3. - PERIMETER PARAMETER OF THE ELLIPSE BOXWING

$\frac{H}{\eta_1}$	$\frac{\sigma_e}{\eta_1}$	$\frac{H}{\eta_1}$	$\frac{\sigma_e}{\eta_1}$	$\frac{H}{\eta_1}$	$\frac{\sigma_e}{\eta_1}$	$\frac{H}{\eta_1}$	$\frac{\sigma_e}{\eta_1}$	$\frac{H}{\eta_1}$	$\frac{\sigma_e}{\eta_1}$	$\frac{H}{\eta_1}$	$\frac{\sigma_e}{\eta_1}$
0	1.000000	.20	1.050502	.40	1.150656	.60	1.276350	.80	1.418083	1.0	1.570796
.01	1.000275	.21	1.054653	.41	1.156442	.61	1.283110	.81	1.425490	1.2	1.73145
.02	1.000960	.22	1.058912	.42	1.162288	.62	1.289908	.82	1.432922	1.4	1.89808
.03	1.001977	.23	1.063276	.43	1.168192	.63	1.296744	.83	1.440381	1.5	1.98318
.04	1.003286	.24	1.067740	.44	1.174155	.64	1.303617	.84	1.447866	1.6	2.06931
.05	1.004856	.25	1.072303	.45	1.180724	.65	1.310526	.85	1.455376	1.8	2.24423
.06	1.006667	.26	1.076960	.46	1.186245	.66	1.317471	.86	1.462910	2.0	2.42211
.07	1.008700	.27	1.081709	.47	1.192371	.67	1.324450	.87	1.470470	2.5	2.54309
.08	1.010940	.28	1.086547	.48	1.198549	.68	1.331464	.88	1.478053	3.0	3.34118
.09	1.013375	.29	1.091470	.49	1.204778	.69	1.338512	.89	1.485660	3.5	3.81272
.10	1.015994	.30	1.096478	.50	1.211056	.70	1.345592	.90	1.493290	4	4.28921
.11	1.018786	.31	1.101567	.51	1.217383	.71	1.352705	.91	1.500943	5	5.25251
.12	1.021742	.32	1.106734	.52	1.223758	.72	1.359850	.92	1.508619	6	6.22502
.13	1.024856	.33	1.111977	.53	1.230179	.73	1.367025	.93	1.516317	8	8.18524
.14	1.028119	.34	1.119741	.54	1.236658	.74	1.374232	.94	1.524037	10	10.15994
.15	1.031525	.35	1.122685	.55	1.243157	.75	1.381468	.95	1.531778	20	20.09713
.16	1.035068	.36	1.128146	.56	1.249712	.76	1.388734	.96	1.539541	30	30.07148
.17	1.038742	.37	1.133675	.57	1.256310	.77	1.396029	.97	1.547324	40	40.05720
.18	1.042543	.38	1.139271	.58	1.262949	.78	1.403353	.98	1.555128	50	50.04800
.19	1.046464	.39	1.144932	.59	1.269630	.79	1.410704	.99	1.562952	∞	$\rightarrow H/\eta_1$

APPLICATION TO PARTIAL SPAN BOXWINGS

When a strut is designed so that it contributes to the lift of the vehicle and since the strut is attached to the wing, the configuration is the same as that of the boxwing, in particularly, partial span boxwings. One objective of this section is to demonstrate usage of the generalized equation (17) for estimating e for this type of wing.

For the rectangle boxwing, $H_a = H$, $\eta_0 = 0$, $\eta_1 = \eta_{10} = \eta_a = 1$, $\sigma = 1 + H$, and equations (17) and (21) reduce to

$$\left.\begin{aligned} e &= (1 + \tfrac{4}{\pi} H)(1 + f) \\ P &= \sigma - \sigma_e \end{aligned}\right\} \qquad (33)$$

Equation (33) is the same as equation (10) which is a solution for the rectangle boxwing.

For the ellipse boxwing, $H_a = \pi H/4$, $\eta_a = \pi H/4$, $\sigma = \sigma_e$, then equation (17) becomes $e = 1 + H$, which is equal to equation (6).

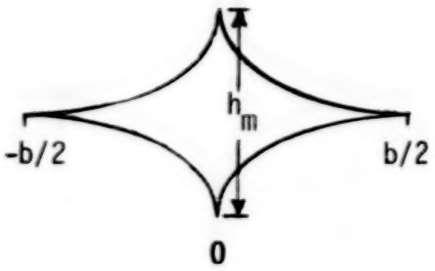
For the diamond boxwing, $\eta_a = 1/2$, $H_a = H/2$, $\eta_0 = 0$, $\eta_1 = 1$, $\eta_{10} = 1$, and $\sigma = \sqrt{1 + H^2}$, then equations (17) with (18) and (21) reduce to

$$e = \left[1 + \frac{2}{\pi} H \left(1 - .44604 \frac{1 - .19H}{1 + 1.81H} \frac{1 + .4826H^2}{1 + .255H^{5/2}}\right)\right](1 + f) \qquad (34)$$

$$P = .35355 \left(\sqrt{1 + H^2} - \sigma_e\right) \qquad (35)$$

where with the P of equation (35), f is determined from equation (20). These f values are negative since P is negative. Values of e computed from equation (34) compare within one-tenth of one percent accuracy with those of the diamond boxwing listed in table 1.

Reflexed Ellipse Boxwing



For this wing, $\eta_a = \eta_{af} = 1 - \pi/4$, $H_a/H = 1 - \pi/4$, $\eta_{10} = 1$, $\sigma = \sigma_e$, then $P = 0$, $f = 0$, and equation (17) becomes

$$e = 1 + \left(\frac{4}{\pi} - 1\right)H\left[1 - \left(5 - \frac{\pi}{2} - \frac{8}{\pi}\right)G_0\right]$$

$$= 1 + .27324H\,(1 - .881725\,G_0) \qquad (36)$$

where G_0 is given in equation (18) with $\eta_{10} = 1$. Some typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.00721	1.02008	1.07519	1.19110	1.46881

Partial Span Ellipse Boxwing

For this wing, $\eta_a = \eta_{af} = (\pi/4)\eta_1$, $H_a/H = (\pi/4)\eta_1$, $\eta_{10} = \eta_1$, η_1 is a specified value, $0 \le \eta_1 \le 1$, $\sigma = \sigma_e$, then $P = 0$, $f = 0$, and equation (17) becomes

$$e = 1 + \eta_1 H\left[1 - (1 - \eta_1^2)\left(1 - \frac{\pi^2}{16}\eta_1^2\right)G_0\right] \qquad (37)$$

When $H = \eta_1$, the central wing is circular. For $\eta_1 = 0.5$, the equation and some values of e are

$$e = 1 + .5H\,(1 - .63434\,G_0)$$

where G_0 is given in equation (18) with $\eta_{10} = 0.5$.

H	0	.1	.2	.5	1	2
e	1	1.02116	1.04710	1.14372	1.33842	1.78399

Partial Span Semi-Ellipse Boxwing

For this wing, $\eta_a = \eta_{af} = (\pi/4)\eta_1$, $H_a/H = (\pi/4)\eta_1$, $\eta_{10} = \eta_1$, then equations (17) and (21) become

$$e = \left\{1 + \eta_1 H \left[1 - (1 - \eta_1^2)(1 - \frac{\pi^2}{16}\eta_1^2)G_0\right]\right\}(1 + f) \tag{38}$$

$$P = \left[\frac{\frac{\pi}{4}\eta_1^2}{1 + 3(1 - \frac{\pi}{4}\eta_1^2)(1 - \frac{\pi}{2}\eta_1^2)^2}\right]^{3/2} (\sigma - \sigma_e) \tag{39}$$

The semi-ellipse perimeter parameter is

$$\sigma = \frac{\eta_1}{2}\left\{1 + \begin{bmatrix} E, & k = \sqrt{1 - (\frac{2H}{\eta_1})^2}, & H \le \frac{\eta_1}{2} \\ \frac{2H}{\eta_1}E, & k = \sqrt{1 - (\frac{2H}{\eta_1})^{-2}}, & H \ge \frac{\eta_1}{2} \end{bmatrix}\right\} \tag{40}$$

and σ_e is given in equation (24). With $\eta_1 = 0.5$, equations (38) through (40) reduce to

$$\left.\begin{aligned} e &= \left[1 + \frac{H}{2}(1 - .63434\, G_0)\right](1 + f) \\ P &= .033507\,(\sigma - \sigma_e) \\ \sigma &= \frac{1}{4} + \frac{1}{4}\begin{bmatrix} E, & k = \sqrt{1 - (4H)^2}, & H \le \frac{1}{4} \\ 4HE, & k = \sqrt{1 - (4H)^{-2}}, & H \ge \frac{1}{4} \end{bmatrix} \end{aligned}\right\} \tag{41}$$

Using equation (41) with equations (18) and (22), for $\eta_1 = 0.5$, some typical values for e are

H	0	.1	.2	.5	1	2
e	1	1.02324	1.05142	1.15340	1.35489	1.81216

For the full-span semi-ellipse boxwing, $\eta_1 = 1$, then equations (38) through (40) reduce to

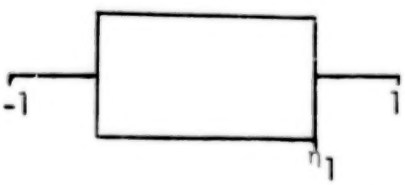
$$\left.\begin{aligned} e &= (1 + H)(1 + f) \\ P &= .523103\ (\sigma - \sigma_e) \\ \sigma &= \frac{1}{2} + \frac{1}{2}\left[\begin{array}{ll} E, & k = \sqrt{1 - (2H)^2},\ H < \frac{1}{2} \\ 2HE, & k = \sqrt{1 - (2H)^{-2}},\ H > \frac{1}{2} \end{array}\right] \end{aligned}\right\} \quad (42)$$

Using equation (42), for $\eta_1 = 1$, some typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.11669	1.23959	1.61191	2.22938	3.45462

Partial Span Rectangle Boxwing

For this wing, $\eta_a = \eta_{af} = \eta_1$, $H_a/H = \eta_1$, $\sigma = \eta_1 + H$, $\eta_{10} = \eta_1$, then equations (17) and (21) become



$$e = \left\{1 + \frac{4\eta_1}{\pi}\left[1 - \left(1 - \frac{16}{\pi^2}\eta_1^2\right)\left(1 - \eta_1^2\right) G_0\right]\right\}(1 + f) \quad (43)$$

$$P = \left[\frac{\eta_1^2}{1 + 3(1 - \eta_1^2)(1 - 2\eta_1^2)^2}\right]^{3/2} (\eta_1 + H - \sigma_e) \quad (44)$$

With $\eta_1 = 0.5$, equations (43) and (44) reduce to

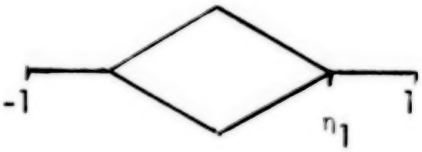
$$\left.\begin{aligned} e &= \left[1 + \frac{2H}{\pi}(1 - .44604\, G_0)\right](1 + f) \\ P &= .064\,(H + .5 - \sigma_e) \end{aligned}\right\} \qquad (45)$$

where σ_e is determined from equation (22) with $\eta_1 = 0.5$. For $\eta_1 = 0.5$, some typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.05343	1.10433	1.26529	1.55646	2.18486

Partial Span Diamond Boxwing

For this wing, $\eta_a = \eta_{af} = (1/2)\eta_1$, $H_a/H = (1/2)\eta_1$, $\sigma = \sqrt{\eta_1^2 + H^2}$, $\eta_{10} = \eta_1$, then equations (17) and (21) become



$$e = \left\{1 + \frac{2\eta_1 H}{\pi}\left[1 - (1 - \frac{4}{\pi^2}\eta_1^2)(1 - \frac{1}{4}\eta_1^2)G_0\right]\right\}(1 + f) \qquad (46)$$

$$P = \left[\frac{\frac{1}{2}\eta_1^2}{1 + 3(1 - \frac{1}{2}\eta_1^2)(1 - \eta_1^2)^2}\right]^{3/2} (\sqrt{\eta_1^2 + H^2} - \sigma_e) \qquad (47)$$

With $\eta_1 = 0.5$

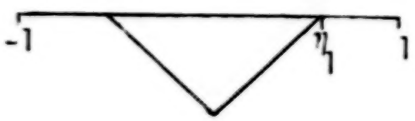
$$e = \left[1 + \frac{H}{\pi}(1 - .84251\, G_0)\right](1 + f) \qquad (48)$$

$$P = .011339\,(\sqrt{.25 + H^2} - \sigma_e) \qquad (49)$$

For $\eta_1 = 0.5$, some typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.00645	1.01685	1.06521	1.17650	1.44816

Partial Span Right Triangle Boxwing



For this wing, $\eta_a = \eta_{af} = (1/2)\,\eta_1$, $H_a/H = (1/2)\,\eta_1$, $\sigma = (1/2)\,\eta_1 + \sqrt{(1/4)\,\eta_1^2 + H^2}$, $\eta_{10} = \eta_1$, then equations (17) and (21) become

$$e = \left\{1 + \frac{2\eta_1 H}{\pi}\left[1 - (1 - \frac{4}{\pi^2}\,\eta_1^2)(1 - \frac{1}{4}\,\eta_1^2)G_0\right]\right\}(1 + f) \tag{50}$$

$$P = \left[\frac{\frac{1}{2}\,\eta_1^2}{1 + 3(1 - \frac{1}{2}\,\eta_1^2)(1 - \eta_1^2)^2}\right]^{3/2} \cdot \left(\frac{1}{2}\,\eta_1 + \sqrt{\frac{1}{4}\,\eta_1^2 + H^2} - \sigma_e\right) \tag{51}$$

For $\eta_1 = 0.5$, typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.00699	1.01853	1.07080	1.18579	1.46190

For $\eta_1 = 1$, typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.03178	1.07181	1.23909	1.60153	2.37390

Ellipse Dumbbell Boxwing

In this wing

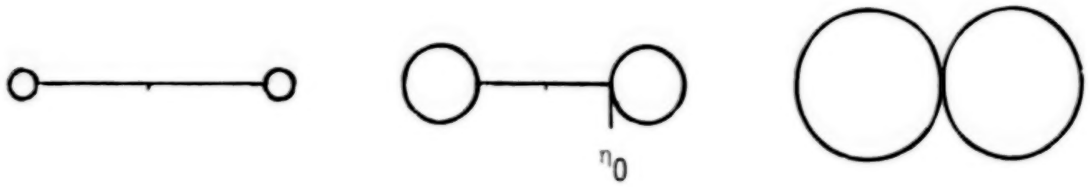
$$\eta_a = \eta_{af} = \frac{1}{2}\,(\eta_1 + \eta_0) + \frac{\pi}{8}\,(\eta_1 - \eta_0)$$

$$\frac{H_a}{H} = \frac{\pi}{4}\,(\eta_1 - \eta_0),\ \eta_{10} = \eta_1 - \eta_0$$

and $\sigma = \sigma_e$, so $P = f = 0$

where σ_e is from equation (24). (52)

With these values and using equations (17) through (19), e can be determined for arbitrary ellipse and position in the wing semispan. For the special case of circular boxwings with $\eta_1 = 1$, the wing appears as shown below.



For the circle dumbbell boxwing with $\eta_1 = 1$, the height parameter is $H = (1 - \eta_0)/2$, and the wing terms simplify to

$$\left.\begin{aligned} \eta_a &= \frac{1}{2}(1 + \eta_0) + \frac{\pi}{8}(1 - \eta_0) \\ \frac{H_a}{H} &= \frac{\pi}{4}(1 - \eta_0) \\ \eta_{10} &= 1 - \eta_0 \\ \eta_{10}\, H &= \frac{1}{2}(1 - \eta_0)^2 \end{aligned}\right\} \qquad (53)$$

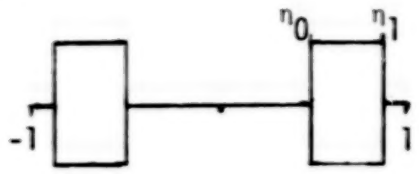
With the values of equation (53) and equations (17) through (19) typical values of e are determined at $\eta_0 = .8, .7, .5$, and 0, which correspond to H values of .1, .15, .25, and .50, respectively.

H	0	.10	.15	.25	.50
e	1	1.17494	1.24393	1.39304	1.64077
ref. 2, e	1	1.175	1.254	1.393	1.630

These circular dumbbell boxwing values compare well with the theory results from the application of the methods of reference 2.

Rectangle Dumbbell Boxwing

In this wing,



$$\left.\begin{aligned} \eta_a &= \eta_{af} = \eta_1 \\ \eta_{10} &= \eta_1 - \eta_0 = \frac{H_a}{H} \\ \sigma &= \eta_1 + H \end{aligned}\right\} \qquad (54)$$

With the terms of equation (54), e is determined by using equations (17) through (21) and (24). For the special case of a square dumbbell boxwing with $\eta_1 = 1$, then $H = (1 - \eta_0)/2$ and the equations simplify to

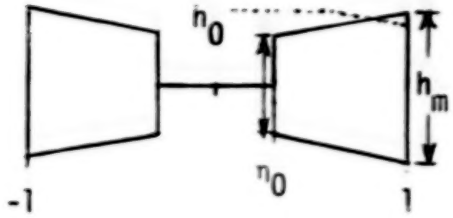
$$\left.\begin{aligned} e &= \left\{1 + \frac{8}{\pi} H^2 \left[1 - \left(1 - \frac{32}{\pi^2} H\right)(1 - 2H) G_0\right]\right\}(1 + f) + \left[1 - (2H)^{1.7}\right](e_e - 1) \\ G_0 &= \frac{(1 - .38\,H^2)(1 + 1.9304\,H^4)}{(1 + 3.62\,H^2)(1 + 1.4425\,H^5)} \\ \sigma &= 1 + H, \quad \sigma_e = 1 + .422112H \\ P &= \frac{8(.577888)H^4}{\left[1 + 3(1 - 4\,H^2)(1 - 8\,H^2)^2\right]^{3/2}} \end{aligned}\right\} \qquad (55)$$

where $H = (1 - \eta_0)/2$. With equation (55), typical values of e of the square dumbbell boxwing are determined at $\eta_0 = .8, .7, .5$, and 0, which correspond to H values of .1, .15, .25, and .50, respectively.

H	0	.10	.15	.25	.50
e	1	1.20451	1.29238	1.50320	2.00026

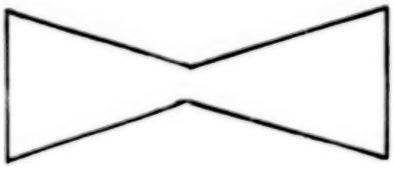
Bow-Tie Boxwing

For this wing,



$$\left.\begin{aligned} &\eta_a = \eta_{af} = 1,\ \frac{H_a}{H} = \frac{1}{2}(1 - \eta_0)\left(1 + \frac{H_0}{H}\right) \\ &\eta_{10} = 1 - \eta_0,\ \sigma = \frac{1}{2}\left[1 + \eta_0 + H + \right. \\ &\left. H_0 + \sqrt{(H - H_0)^2 + (1 + \eta_0)^2}\ \right] \end{aligned}\right\} \qquad (56)$$

With the terms of equation (56), e can be determined by using equations (17) through (21) and (24). For the special case of $\eta_0 = 0$, and $H_0/H = (\pi^2 - 8)/8 = .2337$, the equations for determining e simplify to



$$e = \left(1 + \frac{\pi}{4} H\right)(1 + f) + .560149\ (e_e - 1) \qquad (57)$$

$$P = .44219\left\{\frac{1}{2} + .61685\ H + \frac{1}{2}\sqrt{1 + .58722\ H^2} - \left[\begin{matrix} E,\ k = \sqrt{1 - H^2} \\ HE,\ k = \sqrt{1 - H^{-2}} \end{matrix}\right]\right\}$$

Typical values of e for this bow-tie boxwing computed from equation (57) are

H	0	.1	.2	.5	1	2
e	1	1.24446	1.45077	2.03665	2.94646	4.69481

APPLICATION TO FIN-WINGS

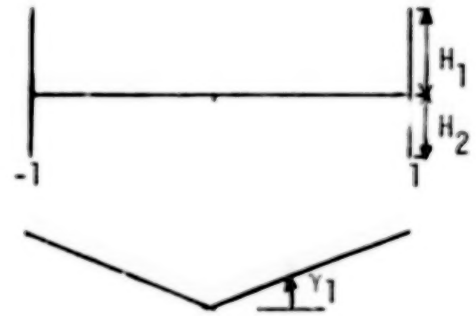
Fin-wing configurations include end-plate wings, winglets, V-wings, cross-wings, and fan wings. Since in this class of wings $H_a = 0$, equation (17) simplifies to

$$e = 1 + G_1 \qquad (58)$$

where G_1 as a function of η_{af} is given in equation (19), and η_{af}, in equation (32). The prediction of e for fin-wings can be demonstrated by application of the method for estimation of η_{af}.

End-Plate Wings and V-Wings

For the end-plate wing



$$\eta_{a1} = \eta_{a2} = \eta_{11} = \eta_{01} = \eta_{13} = 1,$$

$$\gamma_2 = -\gamma_1,$$

then equation (32) reduces to

$$\eta_{af} = \frac{1 + .05 \frac{H_2}{H_1}}{1.05}, \text{ at } H_2 = H_1, \eta_{af} = 1 \qquad (59)$$

At $\eta_{af} = 1$, $G_1 = e_e - 1$, so by equation (58), $e = e_e$, (eq. 16).

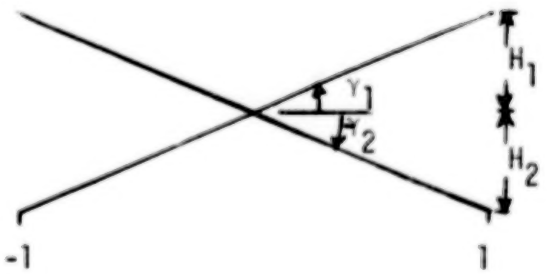
For the V-wing

$$\eta_{a1} = \eta_{a2} = .5, \eta_{11} = 1, \eta_{01} = \eta_{13} = 0, \gamma_2 = \gamma_1,$$

then equation (32) reduces to $\eta_{af} = 1/2$. At this value of η_{af}, $G_1 = e_v - 1$, so by equation (58), $e = e_v$, (eq. 15).

Cruciform or Cross-Wing

For the cross-wing



$$\eta_{a1} = \eta_{a2} = .5, \eta_{11} = \eta_{12} = 1,$$

$$\eta_{01} = \eta_{13} = 0,$$

then equation (32) reduces to

$$\eta_{af} = \frac{1}{2}\left[1 + .257267\sqrt{\frac{1 + H_2^2}{1 + H_1^2}}\left(1 - \frac{\sin \gamma_2}{\sin \gamma_1}\right)\right] \tag{60}$$

When $\gamma_2 = \gamma_1$, then $\eta_{af} = 1/2$, the configuration becomes the V-wing, and as before, $e = e_v$. When $\gamma_2 = -\gamma_1$ and $H_2 = H_1$, the configuration becomes a symmetrical cross-wing, and equation (60) leads to $\eta_{af} = .757267$. With this value of η_{af}, and equation (19), equation (58) gives e for the symmetrical cross-wing as

$$e = 1 + \frac{1.84922\,(e_v - 1)}{\left(.397872 + \sqrt{\frac{e_v - 1}{e_e - 1}}\right)^2} \tag{61}$$

where e_v and e_e are given in equations (15) and (16), and in table 2. Some typical values of e are

H	0	.1	.2	.5	1	2
e	1	1.03997	1.11847	1.43415	2.00000	3.09219

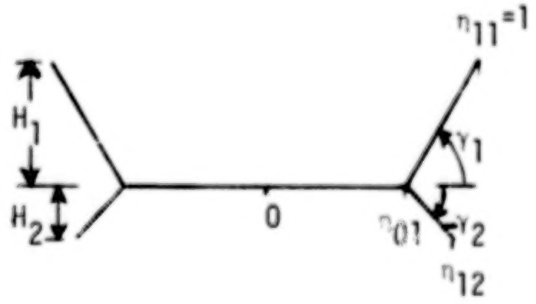
where the H = 1 configuration is a 90 degree symmetrical cross-wing which as shown in appendix D has a value of e = 2.

Winglet Wings

For this wing,

$\eta_{a1} = \frac{1}{2}(1 + \eta_{01})$, $\eta_{a2} = \frac{1}{2}(\eta_{12} + \eta_{01})$,

$\eta_{11} = 1$, $\eta_{13} = \eta_{01}$



then with equation (32)

$$\eta_{af} = \left\{ 1 + \frac{1}{2}\eta_{12}^2 \sqrt{\frac{H_2^2 + (\eta_{12} - \eta_{01})^2}{H_1^2 + (1 - \eta_{01})^2}} \left(1 - \frac{\sin \gamma_2}{\sin \gamma_1}\right) \right.$$

$$\left. \left[.05 + \frac{(1 - \eta_{01})\,|1 - \eta_{01}|^{1/2}}{2.1527} \right] \right\} \frac{1 + \eta_{01}}{2(1 + .05\,\eta_{01}^2)} \qquad (62)$$

For a single winglet, $\gamma_2 = \gamma_1$, then

$$\eta_{af} = \frac{1 + \eta_{01}}{2(1 + .05\,\eta_{01}^2)} \qquad (63)$$

At H = 1, equation (58), with (19) and the e_e and e_v values at H = 1, results in

$$e = 1 + \frac{2.14961\,\eta_{af}^2}{(1 + .12672\sqrt{\eta_{af}})^2} \qquad (64)$$

Typical values of e for various locations of a single winglet for H = 1, are

η_{01}	0	.25	.50	.75	1.00	1.05
η_{af}	.50000	.62305	.74074	.85106	.95238	.97145
e	1.45265	1.68961	1.95892	2.24811	2.54421	2.60315

where the $\eta_{01} = 0$ value is the same as for the V-wing.

For vertically symmetric winglets, $\gamma_2 = -\gamma_1$, $\eta_{12} = 1$, $H_2 = H_1$, then equation (62) becomes

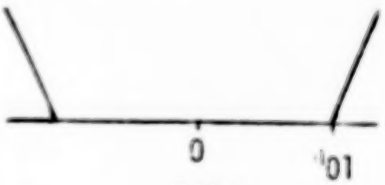
$$\eta_{af} = \left[1.05 + \frac{(1 - \eta_{01})\,|1 - \eta_{01}|^{1/2}}{2.1527} \right] \frac{1 + \eta_{01}}{2(1 + .05\,\eta_{01}^2)} \qquad (65)$$

Then using equation (64), typical values of e for various η_{01} values of a wing with vertically symmetric winglets, for H = 1, are

η_{01}	0	.25	.50	.75	1.00	1.05
η_{af}	.757267	.84219	.89943	.94303	1.00	1.01498
e	2.00000	2.22356	2.38588	2.51568	2.69328	2.74146

where the η_{01} = 0 value is the same as for the symmetric cross-wing of H = 1, and the η_{01} = 1 value is the same as the end-plate wing of H = 1.

For the case of $\gamma_2 = 0$, $\eta_{12} = 1$, then $H_2 = 0$, and equation (62) becomes



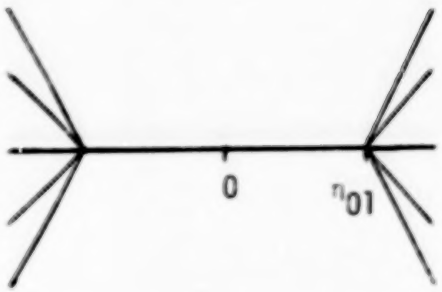
$$\eta_{af} = \left\{1 + \frac{1}{2}\sqrt{\frac{(1-\eta_{01})^2}{H_1^2 + (1-\eta_{01})^2}}\left[.05 + \frac{(1-\eta_{01})\,|1-\eta_{01}|^{1/2}}{2.1527}\right]\right\} \frac{1+\eta_{01}}{2(1+.05\,\eta_{01}^2)} \qquad (66)$$

Then using equation (64), typical values of e for various η_{01} values of a wing with upper surface winglets, for $H = H_1 = 1$, are

η_{01}	0	.25	.50	.75	1.00
η_{af}	.59096	.68880	.77623	.86222	.95238
e	1.62335	1.83499	2.04810	2.27930	2.54421

Fan Wings

To analyze a 2N - 1 fan wing where



$\eta_{1,2n-1} = \eta_{1,2n} = 1$, $\eta_{0,n-1} = \eta_{01}$,

$\gamma_{2n} = -\gamma_{2n-1}$, $H_{2n-1} = H_{2n}$, $\eta_{a,2n} =$

$\eta_{a,2n-1} = (1/2)(1 + \eta_{01})$

then from equation (32)

$$\eta_{af} = \left[1.05 + \frac{(1 - \eta_{01})^{3/2}}{2.1527} + \frac{2}{9}\,\frac{.77 + .9H}{1 + .67H}\,(1 - \eta_{01})^2\right]\frac{1 + \eta_{01}}{2 + .1\,\eta_{01}^2} +$$

$$\frac{.77 + .9H}{1 + .67H}\,\frac{1 - \eta_0^2}{1.15\,N}\left[.05 + \frac{(1 - .36\,\eta_{01})^{3/2}}{2.1527}\right]\sum_{n=3}^{N}\frac{H_{2n-1}}{H_1} \qquad (67)$$

For equally spaced fins along H

$$\left.\begin{aligned}\frac{H_{2n-1}}{H_1} &= \frac{n - 2}{N - 1}\\ \sum_{n=3}^{N}\frac{H_{2n-1}}{H_1} &= \frac{1}{2}(N - 2)\end{aligned}\right\} \qquad (68)$$

For a three-fin fan wing, N = 2, then from the first part of equation (67), the η_{af} values for various η_{01} span stations and typical e values at H = 1 from the equation (64) are

η_{01}	0	.25	.50	.75	1
η_{af}	.86838	.92008	.94059	.95486	1
e	2.29667	2.44668	2.50828	2.55182	2.69328

For other values of H, e can be determined from equation (58) with G_1 given in equation (19), where η_{af} are those listed above.

For a five-fin fan wing, N = 3, and with equations (67) and (68), the η_{af} values and typical e values at H = 1 are

η_{01}	0	.25	.50	.75	1
η_{af}	.94295	.98166	.98352	.97640	1
e	2.51544	2.63513	2.64098	2.61862	2.69328

For an infinite-fin fan wing, N = ∞, then with H = 1

η_{01}	0	.25	.50	.75	1
η_{af}	1.0921	1.1048	1.0694	1.0195	1
e	2.9992	3.0433	2.9216	2.7561	2.6933

The η_{01} = 0 value of e is the same as the bow-tie wing with $\eta_0 = H_0 = 0$, and H = 1. All the above η_{01} = 1 values of e correlate with the e of the symmetric end-plate wing with H = 1.

For the infinite-fin wing with η_{01} = 0, equation (67) simplifies to

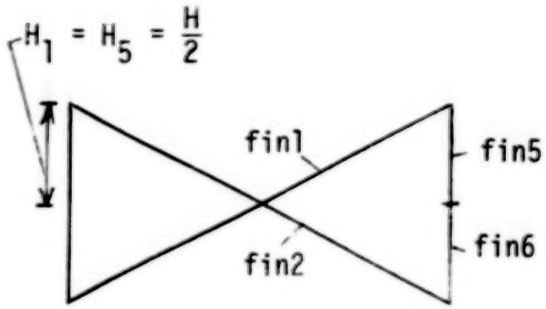
$$\eta_{af} = .75727 + .3348 \frac{.77 + .9H}{1 + .67H} \qquad (69)$$

Then with various H values, typical values of e from equations (58) and (19), are

H	0	.1	.2	.5	1	2
η_{af}	1.01507	1.02712	1.03775	1.06324	1.09207	1.12498
e	1	1.2525	1.4575	2.0677	2.9992	4.7419

These values of e for the infinite-fin wing with $\eta_{01} = 0$, are close to those listed for the bow-tie boxwing where $H_0/H = .2337$, $\eta_0 = 0$. It is apparent the $H_0 = \eta_0 = 0$ bow-tie and infinite-fin cross-wing have the same e.

An example usage of the fin wing equations for other shapes is the case where



$$\eta_{11} = \eta_{12} = 1,\ \gamma_2 = -\gamma_1,\ H_2 = H_1,$$

$$\eta_{01} = \eta_{13} = 0,\ \eta_{15} = \eta_{16} = \eta_{02} = 1,$$

$$\gamma_6 = -\gamma_5 = -\pi/2,\ H_5 = H_6 = H_1,$$

$$\eta_{a5} = \eta_{a6} = 1$$

then equation (32) reduces to

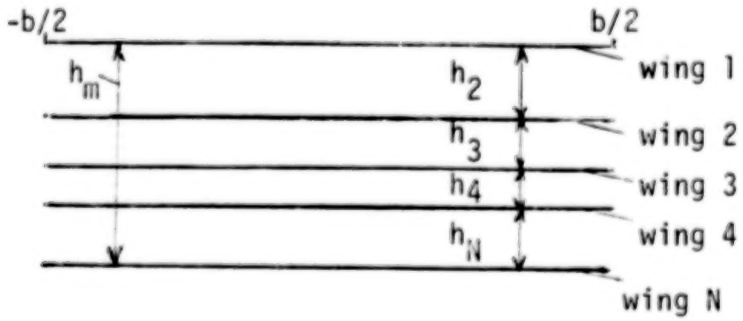
$$\eta_{af} = .75727 + .3344\ \frac{.77 + .9H}{1 + .67H} \qquad (70)$$

Equation (70) is the same as equation (69), hence e is equal to that listed under equation (69).

APPLICATION TO MULTI-WINGS

Multi-Wings with Equal Spans

A vertical stack of wings with equal spans is pictured as



then $H_a = H$, $\eta_a = 1$, and equations (17) and (21) become

$$e = (1 + \frac{4}{\pi} H)(1 + f) \tag{71}$$

$$P = \sigma_M - \sigma_e \tag{72}$$

Comparing equations (71) and (10) shows, for small H so that P and f are small, that the e for multi-wings of equal spans approaches that of the rectangle boxwing. This is in accord with multi-plane thoery. The perimeter p for multi-wings requires a redefinition here since there are no vertical panels. In effect the h distances in p are subdued so that for equal span wings

$$\sigma_M = 1 + \frac{H}{1 + 3.06 \frac{N - 1}{N} H^{1/6} \sum_{n=2}^{N} (\frac{H_n}{H})^2} \tag{73}$$

where σ_M denotes σ of a multi-wing aircraft and N is the number of wings and N - 1 is the number of spaces between wings. For equally vertically spaced wings, $h_2 = h_3 = \ldots h_N$, then

$$\sum_{n=2}^{N} (\frac{H_n}{H})^2 = (N - 1)(\frac{H_2}{H})^2 = \frac{N - 1}{(N - 1)^2} \left[\frac{H_2}{H/(N - 1)}\right]^2 = \frac{1}{N - 1}$$

thus for wings of equal spans which are vertically equally spaced, the effective perimeter parameter is

$$\sigma_M = 1 + \frac{H}{1 + \frac{3.06}{N} H^{1/6}} \tag{74}$$

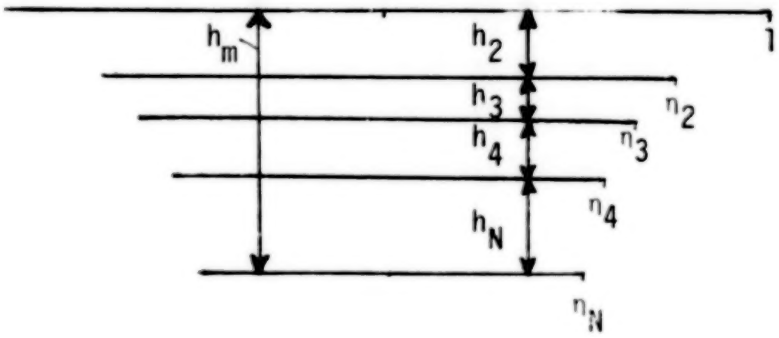
The e is determined by inserting σ_M of equation (73) or (74) into equation (72) and using equations (20) and (71). Equation (71) is accurate in the range $0 \leq H < 1$, but somewhat higher as N becomes large. As mathematically proven in appendix D, in the limit for H very large, the e of the multi-wing and the lifts of wings of equal spans are simply

$$e = N, \text{ and } \frac{L_n}{L_1} = 1 \tag{75}$$

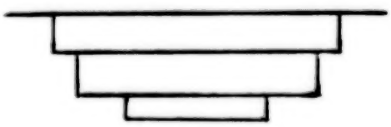
which is independent of H_n distribution. As the number N of wings becomes large, then as shown in equation (74), $\sigma = 1 + H$, which is also the value for the rectangle boxwing. Thus the e of the multi-wing with an infinite number of equal span wings is identical to the rectangle boxwing of the same H.

Multi-Wings with Arbitrary Spans

A vertical stack of wings with various spans is pictured as



The effective integrated value for η_a is the sum of the rectangle areas formed by each of the multiwings, then



$$\eta_a = \frac{H_a}{H} = \sum_{n=2}^{N} \eta_n \frac{H_n}{H}, \text{ where } H = \sum_{n=2}^{N} H_n \tag{76}$$

where $H_n/H = (h_n/b)/(h_m/b)$, and b is the span of the widest wing. The effective perimeter of the multiwing is approximated by

$$\sigma = \frac{\eta_2 + \eta_N}{2} + H_2 + \sum_{n=2}^{N} \sqrt{\left(\frac{\eta_n - \eta_{n+1}}{2}\right)^2 + H_{n+1}^2} \tag{77}$$

The multiwing perimeter parameter is

$$\sigma_M = \eta_2 + \frac{\sqrt{(\sigma - \eta_2)^2 + (1 - \eta_2)^2 B^2} - (1 - \eta_2)B}{1 + B} \tag{78}$$

where

$$B = 3.06 \frac{N - 1}{N} H^{1/6} \sum_{n=2}^{N} (2 - \eta_n)^2 \left(\frac{H_n}{H}\right)^2 \tag{79}$$

As N approaches infinity the multi-wings outline a boxwing and equations (76) and (77) become the values of the boxwings. Then since at $N = \infty$, $B = 0$, the prediction of e for the multi-wing of infinite wings is identical to that of the boxwing formed by the outline of the infinite wings.

The η_{10} parameter is η_2 which represents the span extent of the wing with second largest span, then

$$\eta_{10} = \eta_2 \tag{80}$$

The elliptic σ_e parameter is for a partial span to the η_2 or η_{10} span station. The values for σ_e are determined from using σ_e/η_{10} instead of σ_e/η_1 in equation (22) or in table 2. Thus with η_a and H_a/H from equation (76), and σ_M from equation (78), then P is evaluated from equation (21) which becomes

$$P = \left[\frac{\eta_{10}\eta_a}{1 + 3(1 - \eta_{10}\eta_a)(1 - 2\eta_{10}\eta_a)^2}\right]^{3/2} \left(\sigma_M - \eta_{10} \frac{\sigma_e}{\eta_{10}}\right) \tag{81}$$

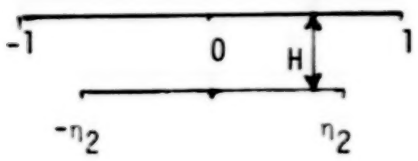
With P, f is found from equation (20), then e computed from equations (17) and (18).

Accuracy is ensured to H values such that $e/(1 + f)$ is the same as the $H = \infty$ value of e. The $H = \infty$ value is derived in appendix D where from equation (D9), for maximum e at $H = \infty$

$$e = 1 + \sum_{n=2}^{N} \eta_n^2, \quad \frac{L_n}{L_1} = \eta_n^2 \tag{82}$$

For finite height parameter H, e and L_n are less than these values.

The biplane solution has N = 2, then



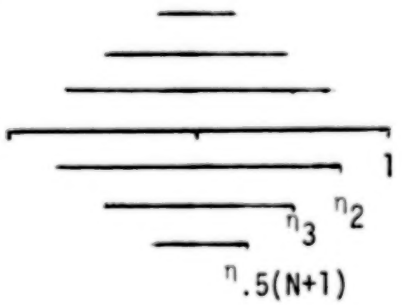
$H_2 = H$, $\eta_a = H_a/H = \eta_{10} = \eta_N = \eta_2$ and equations (77), (78), and (79) become

$$\left.\begin{aligned} \sigma &= \eta_2 + H, \quad B = 1.53(2 - \eta_2)^2 H^{1/6} \\ \sigma_M &= \eta_2 + \frac{\sqrt{H^2 + (1 - \eta_2)^2 B^2} - (1 - \eta_2)B}{1 + B} \end{aligned}\right\} \qquad (83)$$

With these values P is determined from equation (81) and e from equations (17), (18), and (20).

Diamond Shaped Multi-Wings

A diamond shaped multi-wing can be represented by letting wing gaps and wing spans be



$$\left.\begin{aligned} H_n &= \frac{H}{N - 1} \\ \eta_n &= \frac{(1/2)(N + 1) + 1 - n}{(1/2)(N + 1)} \end{aligned}\right\} \qquad (84)$$

where N is an odd integer. Then from equation (76)

$$\eta_a = \frac{H_a}{H} = 2 \sum_{n=2}^{(1/2)(N+1)} \eta_n \frac{H_n}{H} = \frac{4}{N^2 - 1} \sum_{n=2}^{(1/2)(N+1)} \left(\frac{N + 3}{2} - n\right) =$$

$$\frac{4}{N^2 - 1}\left[\left(\frac{N + 1}{2} - 1\right)\left(\frac{N + 3}{2}\right) - \frac{1}{2}\left(\frac{N + 1}{2}\right)\left(\frac{N + 1}{2} + 1\right) + 1\right] = \frac{1}{2} \qquad (85)$$

or for all odd integer N, $\eta_a = H_a/H = 1/2$. The η_{10} value is

$$\eta_{10} = \eta_2 = \frac{N - 1}{N + 1} \tag{86}$$

The perimeter is

$$\sigma = \eta_{\frac{N+1}{2}} + 2H_2 + \sqrt{(\eta_2 - \eta_{\frac{N+1}{2}})^2 + (H - 2H_2)^2}$$

$$= \frac{2}{N + 1} + \frac{2H}{N - 1} + \sqrt{(\frac{N - 3}{N + 1})^2 + (\frac{N - 3}{N - 1})^2 H^2} \tag{87}$$

The multi-wing perimeter parameter σ_M is determined from equation (78) in which η_2 is given by equation (86). B from equation (79) summed for this diamond shape is

$$B = \frac{6.12\ H^{1/6}}{N} \quad \frac{1 + (7/6)N}{1 + N} \tag{88}$$

With σ_M evaluated, P is computed from equation (81) in which $\eta_a = 1/2$, η_{10} is from equation (86), and σ_e is with the ellipse extending to η_{10}.

When N becomes infinite, these parameters become $\eta_a = H_a/H = 1/2$ (for any N), $\eta_{10} = \eta_2 = 1$, B = 0, $\sigma_M = \sigma = \sqrt{1 + H^2}$. These values are the same as those listed in equations (34) and (35) for the diamond boxwing, hence e is the same as those listed in table 1 for the diamond.

Numerical Values of e for Multi-Wings

For equal span wings and vertically equally spaced wings, for the biplane N = 2, and for the triplane N = 3, equation (74) becomes

$$\sigma_M = 1 + \frac{H}{1 + 1.53\ H^{1/6}} \text{ (biplane)}$$

$$= 1 + \frac{H}{1 + 1.02\ H^{1/6}} \text{ (triplane)} \tag{89}$$

With equation (89), P is evaluated from equation (72) with σ_e of the ellipse extending to $\eta_1 = 1$. With P, f is evaluated from equation (20), then e from equation (71). These e's are listed in table 4 for various H values with the $H = \infty$ values from equation (75).

For a partial span wing on the biplane, $N = 2$, $H_2 = H$, $\eta_a = H_a/H = \eta_{10} = \eta_2$, and σ_M is given in equation (83), where η_2 denotes the spanwise extent of the lower wing. Equation (81) becomes

$$P = \left[\frac{\eta_2^2}{1 + 3(1 - \eta_2^2)(1 - 2\eta_2^2)^2}\right]^{3/2} (\sigma_M - \sigma_e) \qquad (90)$$

where σ_e is with the ellipse extending to the $\eta_{10} = \eta_2$ span station. With P, f is determined from equation (20). Equation (17) becomes

$$e = \left\{1 + \frac{4}{\pi}\eta_2 H\left[1 - (1 - \frac{16}{\pi^2}\eta_2^2)(1 - \eta_2^2)G_0\right]\right\}(1 + f) \qquad (91)$$

where G_0 is given in equation (18) in which $\eta_{10} = \eta_2$. The e for the biplane with $\eta_2 = 0.6$ computed from equation (91) is listed in table 4 for various H values. The $H = \infty$ value of e is from equation (82).

TABLE 4. - IDEAL e OF VARIOUS MULTI-WINGS

H	equal span biplane eq. (89)	equal span triplane eq. (89)	$\eta_2 = 0.6$ span biplane eq. (91)
0	1	1	1
.1	1.2021	1.2176	1.0456
.2	1.3529	1.3834	1.0876
.3	1.4810	1.5315	1.1301
.4	1.5842	1.6654	1.1769
.5	1.6407	1.7839	1.2154
∞	2	3	1.36

CONCLUSIONS

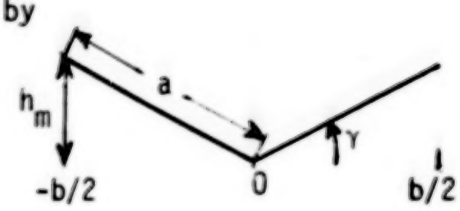
It has been demonstrated that the application of the generalized equations (17) through (32) and equations (76) through (82) provides a direct method for estimates of ideal e for arbitrary lateral-vertical wing forms. Example predictions of e are made for a wide range of wing configurations, from multi-wings to fan and dumbbell wings. Several check points were verified with other methods including the cruciform at H = 1 (eq. 61), the circle dumbbell (eq. 53), and biplane and triplane (table 4 with ref. 1). Also, multi-wing solutions converge to those boxwings as the number of wings increase, and fan wing solutions converge to those of bow-tie wings as the number of fins increase. For the V-wing, an exact closed function was developed. An asymtotic unique maximization solution for multi-wings was found which resulted in exact expressions for ideal e and ideal lift of each wing.

Hampton Technical Center, Kentron International, Inc.
Hampton, Virginia 23666
June 20, 1980

APPENDIX A

V-WING e FACTOR

From reference 3, the added mass M is given by



$$M = \pi\rho \frac{1 - \alpha}{\alpha}$$

where (here γ notation replaces β of ref. 3)

$$\alpha = \frac{1}{2} - \frac{\gamma}{\pi} \tag{A1}$$

The dimensionless added mass is given by

$$\frac{M}{\rho\pi a^2} = \frac{1 - \alpha}{\alpha a^2} \tag{A2}$$

where a is determined from the definite integral

$$a = \int_0^1 \frac{\xi d\xi}{(1 - \xi)^{1-\alpha}(c + \xi)^{\alpha}} \tag{A3}$$

where

$$c = \frac{1 - \alpha}{\alpha}, \quad 0 \leq \alpha \leq \frac{1}{2} \tag{A4}$$

This integral was evaluated numerically in reference 3 for various values of γ, then dimensionless added mass was computed and tabulated. However, as will be shown here, the integration in equation (A3) can be obtained by expanding the second parenthesis term into a binomial series. Thus

$$\frac{\xi}{(c + \xi)^{\alpha}} = \frac{1}{c^{\alpha}}\left[\xi - \frac{\alpha}{c}\xi^2 + \frac{\alpha}{2!c^2}(\alpha + 1)\xi^2 \pm \ldots + \right.$$

$$\left. \frac{(-1)^n \alpha}{n!c^n}(\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 1)\xi^{n+1} + \ldots \right] \tag{A5}$$

The definite integrals of the following type can be evaluated:

$$\int_0^1 \frac{\xi^{n+1}\, d\xi}{(1-\xi)^{1-\alpha}} \begin{matrix} n=0 \\ = \end{matrix} \frac{1}{\alpha(\alpha+1)} \begin{matrix} n=1 \\ = \end{matrix} \frac{2!}{\alpha(\alpha+1)(\alpha+2)}$$

$$\begin{matrix} n=2 \\ = \end{matrix} \frac{3!}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$\begin{matrix} n=n \\ = \end{matrix} \frac{(n+1)!}{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n+1)} \qquad \text{(A6)}$$

Inserting equation (A5) into equation (A3) and applying equation (A6) leads to

$$a = \frac{1}{c^{\alpha}}\left[\frac{1}{\alpha(\alpha+1)} - \frac{2}{(\alpha+1)(\alpha+2)c} + \frac{3}{(\alpha+2)(\alpha+3)c^2} - \ldots + \right.$$

$$\left. \frac{(-1)^n(n+1)}{(\alpha+n)(\alpha+n+1)c^n} + \ldots \right] \qquad \text{(A7)}$$

With c given in equation (A4), equation (A7) can be written as

$$a = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{(n+\alpha)(n+1+\alpha)} \left(\frac{\alpha}{1-\alpha}\right)^{n+1} \qquad \text{(A8)}$$

For the range of $0 \leq \alpha \leq 1/2$, this summation is identically unity. Thus the exact solution for a is simply

$$a = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \qquad \text{(A9)}$$

With equation (A2) the dimensionless added mass is

$$\frac{M}{\rho \pi a^2} = \left(\frac{\alpha}{1-\alpha}\right)^{1-2\alpha} \qquad \text{(A10)}$$

From reference 3, the induced drag is given by

$$D_i = \frac{L^2}{2MV^2} \tag{A11}$$

which equated to equation (1) results in

$$e = \frac{4a^2}{b^2}\left(\frac{M}{\rho\pi a^2}\right) \tag{A12}$$

From the V-wing sketch the geometrical a and γ are

$$\frac{2a}{b} = \frac{1}{\cos\gamma} = (1 + 4H^2)^{1/2} \quad , \gamma = \tan^{-1} 2H \tag{A13}$$

With equation (A1), the dimensionless added mass and factor e, in terms of the γ angle in radians, in closed functions, are

$$\frac{M}{\rho\pi a^2} = \left(\frac{1 - \frac{2\gamma}{\pi}}{1 + \frac{2\gamma}{\pi}}\right)^{\frac{2\gamma}{\pi}} \tag{A14}$$

$$e = \frac{1}{\cos^2\gamma}\left(\frac{1 - \frac{2\gamma}{\pi}}{1 + \frac{2\gamma}{\pi}}\right)^{\frac{2\gamma}{\pi}} \tag{A15}$$

In terms of the height parameter, $H = h_m/b$

$$\frac{M}{\rho\pi a^2} = \left(\frac{1 - \frac{2}{\pi}\tan^{-1} 2H}{1 + \frac{2}{\pi}\tan^{-1} 2H}\right)^{\frac{2}{\pi}\tan^{-1} 2H} \tag{A16}$$

$$e = (1 + 4H^2)\left(\frac{1 - \frac{2}{\pi}\tan^{-1} 2H}{1 + \frac{2}{\pi}\tan^{-1} 2H}\right)^{\frac{2}{\pi}\tan^{-1} 2H} \tag{A17}$$

Equations (A15) and (A17) give e in terms of elementary functions of either the dihedral angle or the height parameters of the V-wing.

APPENDIX B

END-PLATE WING e FACTOR

From page 211 of reference 1,

$$\gamma = \beta k \ \mathrm{sn}\ \eta, \ \frac{\gamma}{\beta} = \left(\frac{K - E}{K}\right)^{1/2} \tag{B1}$$

then

$$\mathrm{sn}\ \eta = \frac{1}{k}\left(\frac{K - E}{K}\right)^{1/2} = \sin\phi \tag{B2}$$

From the definition of snη related to sinϕ, η is

$$\eta = F(\phi \backslash \alpha) = \int_0^{\phi} \frac{d\phi}{(1 - \sin^2\alpha \sin^2\phi)^{1/2}} \tag{B3}$$

that is, the incomplete elliptic integral of the first kind. In addition, from reference 1,

$$\left.\begin{aligned} h_m &= 2\beta\left[E(\eta) - \left(1 - \frac{\gamma^2}{\beta^2}\right)\eta\right], \ b = \frac{\beta\pi}{k}, \\ e &= \frac{4\beta^2}{b^2}\left(\frac{2E}{K} - k'^2\right) \end{aligned}\right\} \tag{B4}$$

where $k = \sin\alpha$, $k' = (1 - k^2)^{1/2} = \cos\alpha$. With equations (B1), (B2), (B3), the h_m of equation (B4) becomes

$$H = \frac{2\gamma}{\pi}\left[E(\phi \backslash \alpha) - \frac{E}{K} F(\phi \backslash \alpha)\right] \tag{B5}$$

$$\phi = \sin^{-1}\left[\frac{1}{\sin\alpha}\left(\frac{K - E}{K}\right)^{1/2}\right] \tag{B6}$$

where $F(\phi \backslash \alpha)$ and $E(\phi \backslash \alpha)$ are incomplete elliptic integrals of the first and second kinds with amplitude ϕ and modular angle α, and K and E are complete elliptic integrals of first and second kind with modular angle α. The term

within the brackets of equation (B5) is the Jacobian Zeta Function. Then

$$H = \frac{2}{\pi} KZ(\phi \backslash \alpha) \tag{B7}$$

where the function $KZ(\phi \backslash \alpha)$ is tabulated in reference 4 for various values of ϕ and α, and has a maximum value at the ϕ given in equation (B6). From equation (B4)

$$e = \frac{4}{\pi^2} K^2 \left(\frac{2E}{K} - \cos^2 \alpha\right) \tag{B8}$$

As can be seen from equations (B5) through (B8), the factor e is not a direct function of H but is related to H only through the parameter α. That is, a value for α is specified, then ϕ is determined from equation (B6), then H from either equation (B5) or (B7), and e from equation (B8). Some typical results for a range of α values are shown as follows:

α deg	H	e	r	α deg	H	e	r
30	.07194	1.14138	.6996	80	.89782	2.53705	.7312
45	.17351	1.33322	.7082	82	1.0165	2.71842	.7296
60	.34824	1.64573	.7200	84	1.1718	2.95205	.7263
68	.49550	1.89566	.7268	85	1.2713	3.10002	.7233
75	.68710	2.20712	.7318	88	1.7821	3.84313	.7048

This list includes the parameter r which is evaluated from

$$r = \frac{1 + 2H - e}{(e - 1 - \frac{4}{\pi} H)H} \tag{B9}$$

Equations (B5) through (B8) are quite complicated functions and do not permit e predictions for a specified value of H. It has been found that a relatively simple function exists between e and H. In the limit as $\alpha \to 0$ equations (B7) and (B8) become

$$H \to \frac{1}{4} k^2, \quad e \to 1 + \frac{1}{2} k^2$$

where $k = \sin\alpha \to 0$. Eliminating k^2 results in

$$e \overset{H\to 0}{=} 1 + 2H \tag{B10}$$

For $\alpha \to \pi/2$, equations (B7) and (B8) become

$$H \to \frac{2}{\pi} \ell n \frac{4}{k'} , \quad e \to \frac{8}{\pi^2} \ell n \frac{4}{k'}$$

where $k' = (1 - k^2)^{1/2} = \cos\alpha$. Eliminating the log term, e becomes

$$e \overset{H\to\infty}{=} \frac{4}{\pi} H \tag{B11}$$

A combination of equations (B10) and (B11) indicates a function of the following type

$$e = 1 + \left(\frac{4}{\pi} + \frac{2 - \frac{4}{\pi}}{1 + rH}\right) H \tag{B12}$$

where r is a quasi-constant. The solution of equation (B12) for r is equation (B9). As can be seen from the listed H and r values, r is nearly constant with H, but remains a weak function of H. An approximation for r which duplicates those in the list is given by

$$r = .732 - \frac{.0612\ (.8 - H)^2}{1 + .3688\ H^2} \tag{B13}$$

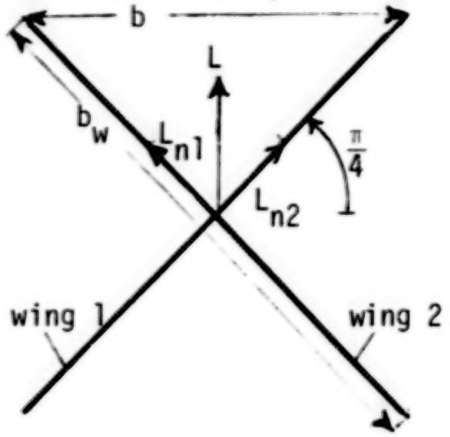
Then equation (B12) gives e with a four to five digit accuracy for the wing with end plates as

$$e = 1 + \left[\frac{4}{\pi} + \frac{2 - \frac{4}{\pi}}{1 + .732\ H - \frac{.0612\ H(.8 - H)^2}{1 + .3688\ H^2}}\right] H \tag{B14}$$

APPENDIX C

90° CRUCIFORM WING

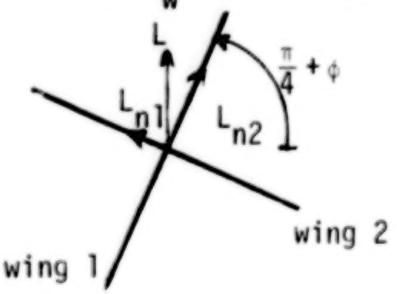
As discussed in reference 5, page 39, in a 90° cruciform wing there is no induced velocity at one vorticity sheet due to the other. In this case each wing acts independently of the other, thus for minimum induced drag the spanwise loading distribution is elliptical along b_w. With L_{n1} and L_{n2} representing the lift normal to wings 1 and 2, with equation (1)



$$D_i = \frac{L_{n1}^2}{\pi q\, b_w^2} + \frac{L_{n2}^2}{\pi q\, b_w^2} = \frac{L^2}{\pi q\, b_w^2} \qquad \text{(C1)}$$

since $L_{n1}^2 + L_{n2}^2 = L^2$. Thus for this case $e = 1$. In reference 5 when the C_L, C_{D_i}, and A are reduced to dimensional quantities with b_w, the solution is also $e = 1$.

For a 90° cruciform wing banked at angle ϕ, the loading is elliptical along wings 1 and 2, and, similarly to equation (C1), the induced drag is



$$D_i = \frac{L_{n1}^2}{\pi q\, b_w^2} + \frac{L_{n2}^2}{\pi q\, b_w^2} = \frac{L^2}{\pi q\, b_w^2} \qquad \text{(C2)}$$

since again $L_{n1}^2 + L_{n2}^2 = L^2$, and hence $e = 1$. Thus when the induced drag of a 90° cruciform is in terms of the wing width, b_w, the efficiency factor is unity at any bank angle.

Generally the induced drag of a lifting configuration is in terms of the lateral span of the configuration, that is, b. Then from equation (1), the

induced drag of the 90° cruciform wing is

$$D_i = \frac{L^2}{\pi q\, b^2\, e} \tag{C3}$$

Equating equations (C3) and (C2) results in

$$e = \left(\frac{b_w}{b}\right)^2 = \frac{1}{\cos^2\left(\frac{\pi}{4} - |\phi|\right)} \tag{C4}$$

where ϕ is the angle of bank of the 90° cruciform wing. When $\phi = 0, \pm\pi/8, \pm\pi/4$, then e = 2, 1.176, and 1, respectively. In summary for the 90° cruciform wing

e = 1, for all bank angles when D_i is defined by equation (C1),

e = equation (C4), for all bank angles when D_i is defined by equation (C3).

APPENDIX D

ASYMPTOTIC SOLUTION FOR OPTIMUM e AND LIFTS FOR MULTI-WINGS WITH LARGE GAPS

If the gap height h_n of a multi-wing is large enough such that the influence of one wing on another is negligible, then the ideal loading for each wing will be elliptic and induced drag is given by $D_{in} = L_n^2/\pi q b_n^2$. The sum of these induced drags equals that of the multi-wing having the sum of all the lifts. Thus for a multi-wing with N planar wings

$$D_i = \frac{L^2}{\pi q\, b_1^2\, e} = \sum_{n=1}^{N} \frac{L_n^2}{\pi q\, b_n^2}\,, \quad L = \sum_{n=1}^{N} L_n \tag{D1}$$

where b_1 is the wing with maximum span. In dimensionless coordinates,

$$\ell_n = \frac{L_n}{L_1}\,, \quad \eta_n = \frac{b_n}{b_1} \tag{D2}$$

then equation (D1) becomes

$$\frac{\left(1 + \sum_{n=2}^{N} \ell_n\right)^2}{e} = 1 + \sum_{n=2}^{N} \frac{\ell_n^2}{\eta_n^2}$$

solving for e

$$e = \frac{\left(1 + \sum_{n=2}^{N} \ell_n\right)^2}{1 + \sum_{n=2}^{N} \frac{\ell_n^2}{\eta_n^2}} \tag{D3}$$

Equation (D3) is valid for any lift variation among the wings, however, there is some lift variation for which e is maximum. For equal loading e can be less than unity when $1 + \eta_2^{-2} + \eta_3^{-2} + \ldots \eta_N^{-2} > N^2$. The maximization technique of reference 5 can be applied to equation (D3). A partial derivative

of e with respect to each of the loading ratios gives

$$\frac{\partial e}{\partial \ell_m} = \frac{2\left(1 + \sum_{n=2}^{N} \ell_n\right)}{\left(1 + \sum_{n=2}^{N} \frac{\ell_n^2}{\eta_n^2}\right)^2} \left[1 + \sum_{n=2}^{N} \frac{\ell_n^2}{\eta_n^2} - \left(1 + \sum_{n=2}^{N} \ell_n\right) \frac{\ell_m}{\eta_m^2}\right]$$

The conditions for maximum e are that these derivatives be zero for each of the loading ratios. Then

$$1 + \sum_{n=2}^{N} \frac{\ell_n^2}{\eta_n^2} - \left(1 + \sum_{n=2}^{N} \ell_n\right) \frac{\ell_m}{\eta_m^2} \equiv 0 \qquad \text{(D4)}$$

These are non-linear simultaneous equations in ℓ_n with n = 2, 3, . . . N unknowns, and m = 2, 3, . . . N equations. For the biplane, N = 2, then equation (D4) with (D3) leads to

$$\left.\begin{aligned} 1 + \frac{\ell_2^2}{\eta_2^2} - (1 + \ell_2)\frac{\ell_2}{\eta_2^2} = 0, \text{ or } \quad & \ell_2 = \eta_2^2 \\ & e = 1 + \eta_2^2 \end{aligned}\right\} \qquad \text{(D5)}$$

For the triplane, N = 3, then the solution of two equations of (D4) leads to

$$\left.\begin{aligned} \ell_2 &= \frac{1 + \frac{\ell_3^2}{\eta_3^2}}{1 + \ell_3}\,\eta_2^2 \\ \frac{\eta_2^2 + \eta_3^2}{\eta_3^4}\,\ell_3^3 &+ \frac{2 - \eta_2^2 - \eta_3^2}{\eta_3^2}\,\ell_3^2 + \frac{1 + \eta_2^2 - 2\eta_3^2}{\eta_3^2}\,\ell_3 - (1 + \eta_2^2) = 0 \end{aligned}\right\} \qquad \text{(D6)}$$

which involves the solution of a cubic equation. As can be seen, the solution of equation (D4) becomes complicated when N ≥ 3. What is needed is to have ℓ_n as some function of η_n so that equation (D4) is satisfied. Such a function has been discovered by the hint given in equation (D5). Let

$$\ell_n = \eta_n^2 \tag{D7}$$

then equation (D4) becomes

$$1 + \sum_{n=2}^{N} \eta_n^2 - \left(1 + \sum_{n=2}^{N} \eta_n^2\right) \frac{\eta_m^2}{\eta_m^2} \equiv 0 \tag{D8}$$

which completely satisfies the maximum conditions, and hence equation (D7) gives the individual wing lifts for maximum e. With equation (D7) inserted into equation (D3) the maximum or ideal e is

$$\ell_n = \eta_n^2, \; e = e_{max} = 1 + \sum_{n=2}^{N} \eta_n^2 \tag{D9}$$

For equal span wings, $\eta_n = 1$, so

$$\ell_n = 1, \; e = N \tag{D10}$$

For biplanes

$$\ell_2 = \eta_2^2, \; e = 1 + \eta_2^2 \tag{D11}$$

For triplanes

$$\ell_2 = \eta_2^2, \; \ell_3 = \eta_3^2, \; e = 1 + \eta_2^2 + \eta_3^2 \tag{D12}$$

and so forth.

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16. Abstract

A relatively simple equation is presented for estimating the induced drag ideal efficiency factor e for arbitrary cross-sectional wing forms. This equation is based on eight basic but varied wing configurations which have exact solutions. The e function which relates the basic wings is developed statistically and is a continuous function of configuration geometry. The basic wing configurations include boxwings shaped as a rectangle, ellipse, and diamond; the V-wing; end-plate wing; 90 degree cruciform; circle dumbbell; and biplane. Example applications of the e equations are made to many wing forms such as wings with struts which form partial span rectangle dumbbell wings; bowtie, cruciform, winglet, and fan wings; and multi-wings. Derivations are presented in the appendices of exact closed form solutions found of e for the V-wing and 90 degree cruciform wing and for an asymptotic solution for multi-wings.

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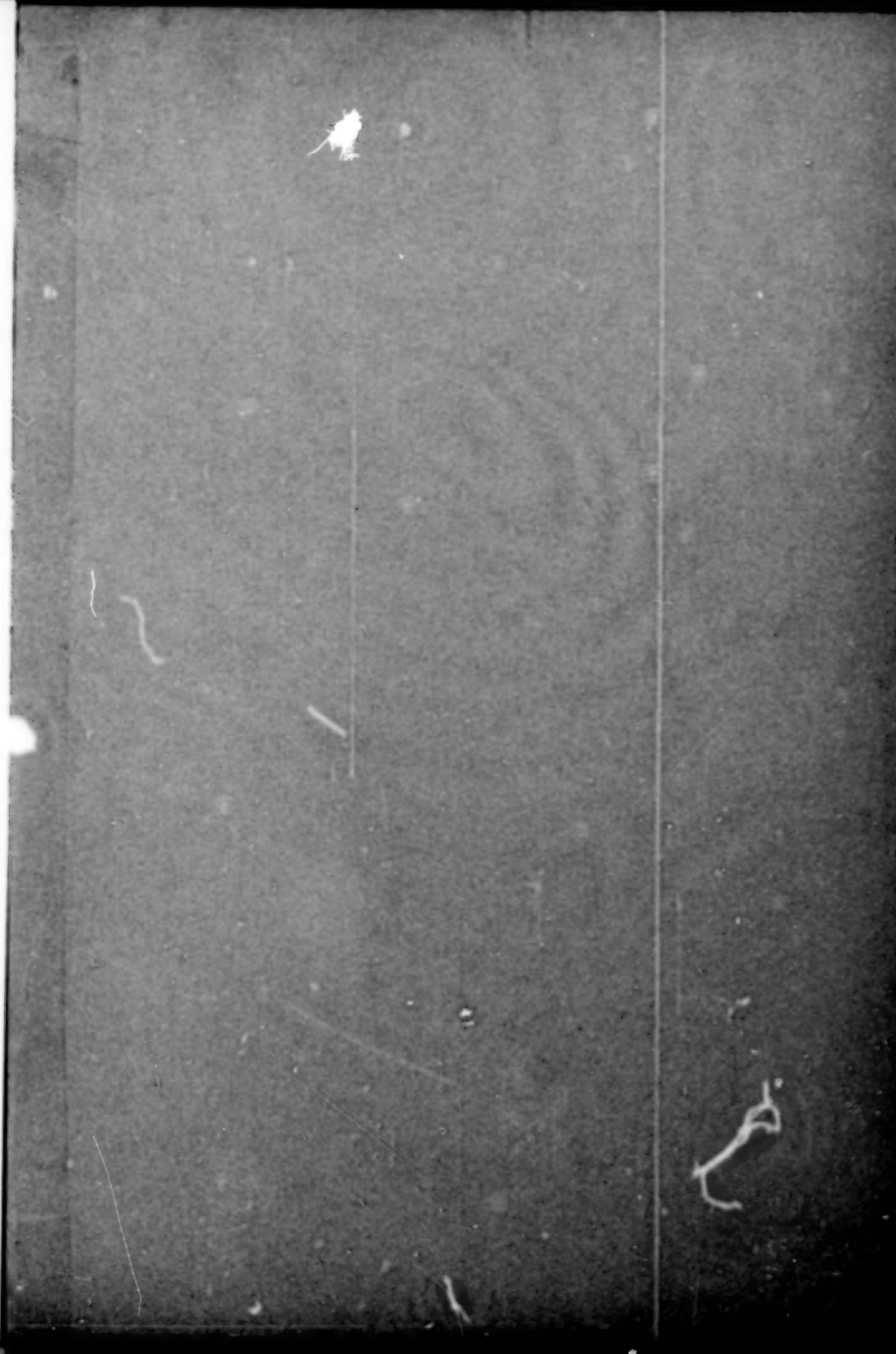

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